

# New solutions in 3D gravity

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## Abstract

We study gravitational theory in  $1 + 2$  spacetime dimensions which is determined by the Lagrangian constructed as a sum of the Einstein-Hilbert term plus the two (translational and rotational) gravitational Chern-Simons terms. When the corresponding coupling constants vanish, we are left with the purely Einstein theory of gravity. We obtain new exact solutions for the gravitational field equations with the nontrivial material sources. Special attention is paid to plane-fronted gravitational waves (in case of the Maxwell field source) and to the circularly symmetric as well as the anisotropic cosmological solutions which arise for the ideal fluid matter source.

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## I. INTRODUCTION

Gravity in a three-dimensional spacetime attracts a lot of attention recently, see [1–6] and the references therein. In a quite general setting, the gravitational field is described in terms of the coframe 1-form  $\vartheta^\alpha$  and the linear connection 1-form  $\Gamma_\alpha^\beta$ . These “potentials” give rise to the field strengths which are the 2-forms of torsion  $T^\alpha$  and curvature  $R_\alpha^\beta$ .

In this paper we continue a study of the 3D gravitational model of Mielke-Baekler which was initiated in [4]. Within the Poincaré gauge approach, the Lagrangian of the model reads:

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$$\begin{aligned}
V_{\text{MB}} = & -\frac{\chi}{2\ell} R^{\alpha\beta} \wedge \eta_{\alpha\beta} - \frac{\Lambda}{\ell} \eta + \frac{\theta_{\text{T}}}{2\ell^2} \vartheta^\alpha \wedge T_\alpha \\
& - \frac{\theta_{\text{L}}}{2} \left( \Gamma_\alpha{}^\beta \wedge d\Gamma_\beta{}^\alpha - \frac{2}{3} \Gamma_\alpha{}^\beta \wedge \Gamma_\beta{}^\gamma \wedge \Gamma_\gamma{}^\alpha \right) + L_{\text{mat}}.
\end{aligned} \tag{1}$$

The coupling constants  $\chi, \theta_{\text{T}}, \theta_{\text{L}}$  are dimensionless, and they specify the usual Hilbert-Einstein, the translational, and the rotational Chern-Simons terms, respectively. The coupling constant  $\ell$  has the dimension of length;  $\Lambda$  is the cosmological constant.

Our geometrical notations and conventions are as follows:  $\eta$  is the volume 3-form,  $\eta_\alpha = e_\alpha \lrcorner \eta$ ,  $\eta_{\alpha\beta} = e_\beta \lrcorner \eta_\alpha$ , and finally  $\eta_{\alpha\beta\gamma} = e_\gamma \lrcorner \eta_{\alpha\beta}$  is the totally antisymmetric Levi-Civita tensor. As usual, Greek indices,  $\alpha, \beta, \dots = 0, 1, 2$ , label the (co)frame components, whereas Latin indices,  $i, j, \dots = 0, 1, 2$ , label the components with respect to a local coordinate basis.

In contrast to [4], we consider not only a vacuum situation, but take the nontrivial matter source into account. Main attention is paid to the case when the dynamical spin current of matter is trivial, thus leaving the energy-momentum current as the only source of the gravitational field. In Sec. II, we demonstrate that a generic model (1) for arbitrary coupling constants  $\chi, \theta_{\text{T}}, \theta_{\text{L}}$  (except for some special choices) is equivalent to a 3D Einstein gravitational theory with certain effective energy-momentum current. The latter differs essentially from the original energy-momentum only when  $\theta_{\text{L}} \neq 0$ , otherwise the two currents are merely proportional to each other. The exceptional models are, as a rule, inconsistent when matter is present, imposing unacceptable mathematical and physical constraints on the structure of the energy-momentum current.

Sec. III is devoted to the derivation of new solutions for a nontrivial electromagnetic source. Specifically, we consider the plane-fronted gravitational wave configurations. In Sec. IV we analyze the case of an ideal fluid source. In particular, the circularly symmetric solutions are derived here, some of which can be interpreted as black holes. As another class of exact solutions we obtain anisotropic cosmological models of a Heckmann-Schücking type. Finally, in Sec. V we summarize the results obtained.

## II. EFFECTIVE EINSTEIN THEORY

The gravitational field equations are derived from the variation of (1) with respect to coframe and (Lorentz) connection:

$$\frac{\chi}{2} \eta_{\alpha\beta\gamma} R^{\beta\gamma} + \Lambda \eta_\alpha - \frac{\theta_T}{\ell} T_\alpha = \ell \Sigma_\alpha, \quad (2)$$

$$\frac{\chi}{2} \eta_{\alpha\beta\gamma} T^\gamma - \frac{\theta_T}{2\ell} \vartheta_\alpha \wedge \vartheta_\beta - \theta_L \ell R_{\alpha\beta} = \ell \tau_{\alpha\beta}. \quad (3)$$

The 2-forms of the material energy-momentum and spin currents are defined by the variational derivatives of the Lagrangian  $L_{\text{mat}}$  of matter  $\Sigma_\alpha := \delta L_{\text{mat}} / \delta \vartheta^\alpha$  and  $\tau_{\alpha\beta} := \delta L_{\text{mat}} / \delta \Gamma_\alpha{}^\beta$ , respectively.

The algebraic system (2)-(3) can be resolved with respect to the torsion and the curvature 2-forms:

$$T^\alpha = \frac{2}{\ell} \left( \mathcal{T} \eta^\alpha + \alpha_L \ell^3 \Sigma^\alpha + \beta \ell^2 \frac{1}{2} \eta^{\alpha\beta\gamma} \tau_{\beta\gamma} \right), \quad (4)$$

$$R^{\alpha\beta} = \frac{1}{\ell^2} \left( \mathcal{R} \vartheta^\alpha \wedge \vartheta^\beta + \alpha_T \ell^2 \tau^{\alpha\beta} + \beta \ell^3 \eta^{\alpha\beta\gamma} \Sigma_\gamma \right). \quad (5)$$

The numeric coefficients are constructed from the coupling constants of the model:

$$\mathcal{T} = \frac{-\frac{\theta_T}{2} \chi + \Lambda \ell^2 \theta_L}{\chi^2 + 2\theta_T \theta_L}, \quad \mathcal{R} = -\frac{\theta_T^2 + \chi \Lambda \ell^2}{\chi^2 + 2\theta_T \theta_L}, \quad (6)$$

$$\alpha_L = -\frac{\theta_L}{\chi^2 + 2\theta_T \theta_L}, \quad \alpha_T = -\frac{2\theta_T}{\chi^2 + 2\theta_T \theta_L}, \quad \beta = -\frac{\chi}{\chi^2 + 2\theta_T \theta_L}. \quad (7)$$

Since in our work we will be mainly interested in the electromagnetic field and the ideal (spinless) fluid as the material sources, we now will specialize to the case of the matter without dynamical spin,  $\tau_{\alpha\beta} = 0$ . Then, as we know, the energy-momentum current is symmetric, i.e.,

$$e_\alpha \lrcorner \Sigma^\alpha = *(\vartheta^\alpha \wedge * \Sigma_\alpha) = 0. \quad (8)$$

However, the trace of the energy-momentum

$$\Sigma = *(\vartheta^\alpha \wedge \Sigma_\alpha) = e_\alpha \lrcorner * \Sigma^\alpha \quad (9)$$

is nontrivial, in general.

We now demonstrate that the above system (2)-(3) is equivalent to an effective Einstein equation. In order to derive this, we note that the local connection 1-form splits into a Riemannian and the contortion parts:

$$\Gamma_{\alpha\beta} = \tilde{\Gamma}_{\alpha\beta} + K_{\alpha\beta}, \quad K_{\alpha\beta} = -e_{[\alpha]}T_{\beta]} + \frac{1}{2} (e_{\alpha]}e_{\beta]}T_{\gamma}) \vartheta^{\gamma}. \quad (10)$$

The curvature 2-form then decomposes accordingly:

$$R_{\alpha}{}^{\beta} = \tilde{R}_{\alpha}{}^{\beta} + \tilde{D}K_{\alpha}{}^{\beta} - K_{\alpha}{}^{\gamma} \wedge K_{\gamma}{}^{\beta}. \quad (11)$$

Hereafter we denote by the tilde the geometrical objects and operators constructed with the help of the Riemannian connection. Using (2), we find

$$K_{\alpha\beta} = \mathcal{K} \eta_{\alpha\beta} + 2\alpha_L \ell^2 \eta_{\alpha\beta\gamma} {}^*\Sigma^{\gamma}, \quad \mathcal{K} := \mathcal{T}/\ell - \alpha_L \ell^2 \Sigma. \quad (12)$$

It is important to note that  ${}^{**} = -1$  for *all* forms in a 3-dimensional spacetime with the metric of Lorentzian signature. Then we have

$$\tilde{D}K_{\alpha}{}^{\beta} = -\eta_{\alpha}{}^{\beta} \wedge d\mathcal{K} + 2\alpha_L \ell^2 \eta_{\alpha}{}^{\beta\gamma} \tilde{D}{}^*\Sigma_{\gamma}, \quad (13)$$

$$-K_{\alpha}{}^{\gamma} \wedge K_{\gamma}{}^{\beta} = -(\mathcal{T}^2/\ell^2 - \alpha_L^2 \ell^4 \Sigma^2) \vartheta_{\alpha} \wedge \vartheta^{\beta} + 2\mathcal{K} \alpha_L \ell^2 \eta_{\alpha}{}^{\beta\gamma} \Sigma_{\gamma} - 4\alpha_L^2 \ell^4 {}^*\Sigma_{\alpha} \wedge {}^*\Sigma^{\beta}. \quad (14)$$

In deriving these formulas we used the symmetry of the energy-momentum current.

As a result, each of the field equations (2) and (3) takes the form of the effective Einstein equation in three dimensions:

$$\frac{1}{2} \eta_{\alpha\beta\gamma} \tilde{R}^{\beta\gamma} + \Lambda^{\text{eff}} \eta_{\alpha} = \ell \Sigma_{\alpha}^{\text{eff}}. \quad (15)$$

Here the effective cosmological term is defined by

$$\Lambda^{\text{eff}} = -\frac{\mathcal{R} + \mathcal{T}^2}{\ell^2}, \quad (16)$$

whereas the effective energy-momentum 2-form reads

$$\begin{aligned} \Sigma_{\alpha}^{\text{eff}} &= (2\alpha_L \mathcal{T} - \beta) \Sigma_{\alpha} + \alpha_L \ell \left( \vartheta_{\alpha} \wedge d\Sigma + 2\tilde{D}{}^*\Sigma_{\alpha} \right) \\ &\quad + \alpha_L^2 \ell^3 \left( -\Sigma^2 \eta_{\alpha} - 2\Sigma \Sigma_{\alpha} + 2\eta_{\alpha\beta\gamma} {}^*\Sigma^{\beta} \wedge {}^*\Sigma^{\gamma} \right). \end{aligned} \quad (17)$$

As an important consistency check, we need to verify that the effective energy-momentum is conserved, just like the original current  $\widetilde{D}\Sigma_\alpha = 0$ . Taking the covariant derivative, we find

$$\widetilde{D}\Sigma_\alpha^{\text{eff}} = -2\alpha_L\ell \left[ \widetilde{R}_\alpha{}^\beta \wedge {}^*\Sigma^\beta + \alpha_L\ell^2 d\Sigma \wedge (\Sigma\eta_\alpha + \Sigma_\alpha) + 2\alpha_L\ell^2 \eta_{\alpha\beta\gamma} {}^*\Sigma^\beta \wedge \widetilde{D}{}^*\Sigma^\gamma \right]. \quad (18)$$

Here we used the Ricci identity  $\widetilde{D}\widetilde{D}{}^*\Sigma_\alpha = -\widetilde{R}_\alpha{}^\beta \wedge {}^*\Sigma_\beta$ . It remains now to substitute the curvature 2-form from the field equation (15) and to use (17). The symmetry (8) of the original energy-momentum yields  $\eta_{[\alpha} \wedge {}^*\Sigma_{\beta]} = 0$  and  $\Sigma_{[\alpha} \wedge {}^*\Sigma_{\beta]} = 0$ . Using all this together with the identity  $\Sigma\eta_\alpha + \Sigma_\alpha = -\eta_{\alpha\beta} \wedge {}^*\Sigma^\beta$ , we finally obtain the conservation law

$$\widetilde{D}\Sigma_\alpha^{\text{eff}} = 0. \quad (19)$$

As another consistency check, let us consider the limiting case of  $\theta_T = \theta_L = 0$ . Then (6) and (16) yield  $\Lambda^{\text{eff}} = \Lambda/\chi$ , while (17) reduces to  $\Sigma_\alpha^{\text{eff}} = \Sigma_\alpha/\chi$ . We thus recover the correct Einstein theory in three dimensions (or, equivalently, the model of Witten [7]).

A very close model arises when  $\theta_L = 0$ . Then we find  $\Lambda^{\text{eff}} = \Lambda/\chi + 3\theta_T^2/(4\ell^2\chi^2)$ , and again  $\Sigma_\alpha^{\text{eff}} = \Sigma_\alpha/\chi$ . In other words, when the Hilbert-Einstein Lagrangian is modified only by the translational Chern-Simons term, the resulting dynamics differs from the Einstein theory only via the shifted cosmological constant.

In contrast, the case  $\theta_T = 0$  represents a nontrivial extension of the Einstein theory. The cosmological constant and the Einstein gravitational coupling constant then are replaced by  $(\Lambda/\chi) \rightarrow (\Lambda/\chi)(1 - \Lambda\ell^2\theta_L^2/\chi^3)$  and  $(1/\chi) \rightarrow (1/\chi)(1 - 2\Lambda\ell^2\theta_L^2/\chi^3)$ , respectively. In addition, the effective energy-momentum (17) contains the new terms proportional to  $\theta_L = 0$ .

As we see from (6)-(7), the reduction to the effective Einstein theory takes place for generic models with  $\chi^2 + 2\theta_T\theta_L \neq 0$ . An exceptional case  $\chi^2 + 2\theta_T\theta_L = 0$  should be analyzed separately. Assuming all the coupling constants to be nonzero, we then find that, for example, the translational coupling can be expressed in terms of the Lorentzian one,  $\theta_T = -\chi^2/(2\theta_L)$ . This immediately imposes the algebraic consistency condition on the right-hand sides of the field equations (2) and (3):  $\ell\Sigma_\alpha = (\chi/\theta_L)\eta_{\alpha\beta\gamma}\tau^{\beta\gamma} + \eta_\alpha[\Lambda - \chi^3/(4\ell^2\theta_L^2)]$ . In particular, the energy-momentum turns out to be proportional to a cosmological term

for a matter with the vanishing dynamical spin. The exceptional case includes both models with Lagrangians containing just one Chern-Simons term. For the purely translational Chern-Simons gravity with  $\theta_T \neq 0$  and  $\chi = 0$ ,  $\theta_L = 0$ , the spin must be constant  $\tau_{\alpha\beta} = -(\theta_T/2\ell^2)\vartheta_\alpha \wedge \vartheta_\beta$  and the spinless matter case is ruled out as inconsistent. Analogously, the purely Lorentzian Chern-Simons gravity with  $\theta_L \neq 0$  and  $\chi = 0$ ,  $\theta_T = 0$  is inconsistent for any nontrivial energy-momentum except the cosmological term  $\ell\Sigma_\alpha = \Lambda\eta_\alpha$ .

### III. EXACT SOLUTIONS WITH ELECTROMAGNETIC SOURCE

As a first new solution, we derive the *pp*-wave solution in 3D gravity. Plane-fronted gravitational wave solutions represent an important class of spacetimes in 4 dimensions [8], as well as in higher dimensions [9]. A particular interest to the *pp*-waves is related with their role in the string theory [10]. Recently, the wave solutions were also studied in the metric-affine gravity models [11].

#### A. Geometry of a *pp*-wave

We choose the local coordinates  $(\sigma, \rho, z)$ , and take the line element in the form

$$ds^2 = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta, \quad (20)$$

with the half-null Lorentz metric

$$g_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (21)$$

The components of the coframe 1-form are given by

$$\vartheta^{\widehat{0}} = -d\sigma, \quad \vartheta^{\widehat{1}} = \left(\frac{q}{p}\right)^2 (s d\sigma + d\rho), \quad \vartheta^{\widehat{2}} = \frac{1}{p} dz. \quad (22)$$

The dual frame basis (such that  $e_\alpha \lrcorner \vartheta^\beta = \delta_\alpha^\beta$ ) is easily constructed:

$$e_{\widehat{0}} = -\partial_{\sigma} + s \partial_{\rho}, \quad e_{\widehat{1}} = \left(\frac{p}{q}\right)^2 \partial_{\rho}, \quad e_{\widehat{2}} = p \partial_z. \quad (23)$$

We choose the functions  $p(z), q(z), s(\sigma, \rho, z)$  as follows:

$$p = 1 + \frac{\lambda}{4} z^2, \quad q = 1 - \frac{\lambda}{4} z^2, \quad s = -\frac{\lambda}{2} \rho^2 + \frac{\sqrt{p}}{2q} H(\sigma, z). \quad (24)$$

This ansatz should be compared to the four-dimensional case [12]. Then, the Riemannian curvature 2-form reads:

$$\tilde{R}^{\alpha\beta} = -\lambda \vartheta^{\alpha} \wedge \vartheta^{\beta} - \eta^{\alpha\beta\gamma} k_{\gamma} k^{\lambda} \eta_{\lambda} p(q^2 s'/p)'. \quad (25)$$

Here we denote the null vector  $k_{\alpha} = \delta_{\alpha}^{\widehat{0}}$  with  $k_{\alpha} k^{\alpha} = 0$ . The derivative with respect to the  $z$  coordinate is denoted by the prime ( $'$ ).

The resulting geometry is regular in sense that the polynomial curvature invariants are always constant, irrespectively of the form of the function  $H(\sigma, z)$ . For example, the quadratic invariant of the Riemannian curvature reads

$$\tilde{R}^{\alpha\beta} \wedge \ast \tilde{R}_{\alpha\beta} = 6\lambda^2 \eta. \quad (26)$$

The cubic invariants are proportional to  $\lambda^3$ , and so on.

## B. Electromagnetic source

Let the material source be represented by the energy-momentum of an electromagnetic wave. Taking the potential 1-form

$$A = \varphi(\sigma, z) \vartheta^{\widehat{0}}, \quad (27)$$

we find the electromagnetic field 2-form  $F = dA = \frac{1}{2} F_{\alpha\beta} \vartheta^{\alpha} \wedge \vartheta^{\beta}$  with the tensor components

$$F_{\alpha\beta} = 2n_{[\alpha} k_{\beta]}. \quad (28)$$

Here  $n^{\alpha} k_{\alpha} = 0$ . In terms of the vector potential, the components of the covector  $n$  read  $n_{\alpha} = \delta_{\alpha}^2 p \varphi'$ . The unknown scalar function  $\varphi$  is determined by the Maxwell's equation  $d \ast F = 0$  which for the metric (20)-(22) reduces to the partial differential equation

$$(p\varphi')' = 0. \quad (29)$$

This equation is easily integrated, yielding  $\varphi' = \nu(\sigma)/p$  with an arbitrary function  $\nu(\sigma)$ .

The electromagnetic energy-momentum current 2-form then reads:

$$\Sigma_\alpha = \frac{1}{2} [(e_\alpha \rfloor F) \wedge {}^*F - F(e_\alpha \rfloor {}^*F)] = \nu^2 k_\alpha k^\beta \eta_\beta. \quad (30)$$

Since the vector  $k_\alpha$  is null, we find  $\vartheta^\alpha \wedge \Sigma_\alpha = 0$ . Furthermore, for the same reason it is obvious that

$$\Sigma_\alpha \wedge \Sigma_\beta = 0, \quad \Sigma_\alpha \wedge {}^*\Sigma_\beta = 0, \quad {}^*\Sigma_\alpha \wedge {}^*\Sigma_\beta = 0. \quad (31)$$

Hence  $\eta_{\alpha\beta\gamma} {}^*\Sigma^\beta \wedge {}^*\Sigma^\gamma = 0$ . Finally, the direct computation yields

$$\widetilde{D} {}^*\Sigma_\alpha = -\frac{\lambda z}{q} \Sigma_\alpha. \quad (32)$$

### C. Explicit $pp$ -wave solutions

We now can substitute (25) and the energy-momentum (30) and (32) into the effective Einstein equation (15). The latter yields the algebraic relation  $\lambda = \Lambda^{\text{eff}}$  and the differential equation:

$$\left[ \frac{q^2}{p} \left( \frac{\sqrt{p}}{2q} H \right)' \right]' = \nu^2 \ell \left( \frac{2\alpha_L \mathcal{T} - \beta}{p} - \frac{2\alpha_L \ell \lambda z}{pq} \right). \quad (33)$$

Integration is straightforward and yields the general solution:

$$\begin{aligned} H(\sigma, z) = \frac{1}{\sqrt{p}} \Big\{ & \mu_1(\sigma) q + \mu_2(\sigma) z + 2\nu^2 \ell [(2\alpha_L \mathcal{T} - \beta) z - 2\alpha_L \ell q] (2/\sqrt{\lambda}) \arctan(\sqrt{\lambda} z/2) \\ & - 2\nu^2 \ell [2\alpha_L \ell z + (2\alpha_L \mathcal{T} - \beta) q/\lambda] \ln(|p/q|) \Big\}. \end{aligned} \quad (34)$$

With the two arbitrary functions  $\mu_{1,2}(\sigma)$ , the first two terms above represent the general solution of the homogeneous equation with the vanishing right-hand side in (33).

The resulting configuration turns out to be localized along the  $z$ -coordinate. Namely, for large positive and negative values of  $z \rightarrow \pm\infty$ , the metric function reads  $s = -\lambda\rho^2/2 +$



$\nu_1/2 + \mathcal{O}(1/z)$ , with  $\nu_1 = \mu_1 \mp 4\pi\nu^2\alpha_L\ell^2/\sqrt{\lambda}$ . The geometry is thus asymptotically de Sitter in these limits. The function  $s$  blows up when  $q = 0$ , i.e., at  $z = \pm 2/\sqrt{\lambda}$ , however there is no curvature singularity at these points, cf. (26). All this refers to the case when the effective cosmological constant is positive,  $\Lambda^{\text{eff}} > 0$ .

When  $\Lambda^{\text{eff}} < 0$ , we obtain a different solution with the negative  $\lambda = -|\Lambda^{\text{eff}}|$ :

$$H(\sigma, z) = \frac{1}{\sqrt{p}} \left\{ \mu_1(\sigma) q + \mu_2(\sigma) z + 2(\nu^2\ell/\sqrt{|\lambda|}) [(2\alpha_L\mathcal{T} - \beta) z - 2\alpha_L\ell q] \ln[(q + \sqrt{|\lambda|}z)/p] \right. \\ \left. - 2\nu^2\ell [2\alpha_L\ell z + (2\alpha_L\mathcal{T} - \beta) q/\lambda] \ln(|p/q|) \right\}, \quad (35)$$

where  $\mu_{1,2}(\sigma)$  are again two arbitrary functions. The properties of this solution are rather similar to those of (34). Here we again observe the similar asymptotic behavior  $s = -\lambda\rho^2/2 + \mu_1/2 + \mathcal{O}(1/z)$  for  $z \rightarrow \pm\infty$ . And analogously, the function  $s$  diverges at  $z = \pm 2/\sqrt{|\lambda|}$ , although the curvature invariants are everywhere regular.

Finally, for the vanishing effective cosmological constant  $\Lambda^{\text{eff}} = 0$  we find  $\lambda = 0$ , hence  $p = q = 1$ , and the integration of (33) yields

$$H(\sigma, z) = \mu_1(\sigma) + \mu_2(\sigma) z + \nu^2\ell (2\alpha_L\mathcal{T} - \beta) z^2. \quad (36)$$

In this case all the curvature invariants (algebraic and differential) are trivial.

#### IV. EXACT SOLUTIONS WITH AN IDEAL FLUID

Let us consider now the ideal fluid which is characterized by the energy density  $\varepsilon$ , pressure  $p$  and the 2-form of the flow  $u$ . The energy-momentum current then reads

$$\Sigma_\alpha = p\eta_\alpha - (\varepsilon + p)u e_\alpha]{}^*u. \quad (37)$$

As usual, we impose the normalization condition  $u \wedge {}^*u = \eta$ . It seems worthwhile to note that the covector of velocity is defined by the Hodge dual,  $-{}^*u$ , of the flow 2-form  $u$ . The conservation law  $\widetilde{D}\Sigma_\alpha = 0$  comprises the two well-known balance relations for the energy density and the pressure:

$$u \wedge d\varepsilon + (\varepsilon + p) du = 0, \quad (38)$$

$$dp + {}^*u {}^*(u \wedge dp) + a = 0. \quad (39)$$

Here the fluid acceleration 1-form is defined by  $a := {}^*(u \wedge \widetilde{D}e_\alpha] {}^*u) \vartheta^\alpha$ . It evidently satisfies the “orthogonality” condition  $u \wedge a = 0$ .

Substituting (37) into (17), we find for the last term of the effective energy-momentum:

$$-\Sigma^2 \eta_\alpha - 2\Sigma \Sigma_\alpha + 2\eta_{\alpha\beta\gamma} {}^*\Sigma^\beta \wedge {}^*\Sigma^\gamma = -2\varepsilon \Sigma_\alpha - \varepsilon^2 \eta_\alpha. \quad (40)$$

### A. Stationary rotating configurations

Let us impose the differential condition on the fluid flow:

$$\widetilde{D}e_\alpha] {}^*u = \mu e_\alpha] u. \quad (41)$$

Here  $\mu$  is an arbitrary constant. Multiplying (41) from the left by the coframe  $\vartheta^\alpha$ , we find

$$d {}^*u = -2\mu u. \quad (42)$$

The above conditions essentially constrain the kinematics of the fluid. Directly from (41) we derive that the acceleration vanishes,  $a = 0$ , and hence (42) shows that the vorticity is nontrivial,  $\omega = d {}^*u = -2\mu u$ . On the other hand, taking the exterior differential of (42), we find that the volume expansion is zero,  $du = 0$ . In other words, we have a class of stationary rotating configurations.

Using these kinematic properties in the conservation law (38)-(39), we conclude that the constant energy density and pressure are compatible with the above assumptions:

$$\varepsilon = \varepsilon_0, \quad p = p_0. \quad (43)$$

Using then (41)-(43) in (15), we obtain the Einstein field equation

$$\frac{1}{2} \eta_{\alpha\beta\gamma} \widetilde{R}^{\beta\gamma} + \underline{\Lambda} \eta_\alpha = \ell \underline{\Sigma}_\alpha, \quad (44)$$

for the effective fluid energy-momentum

$$\underline{\Sigma}_\alpha = \underline{p} \eta_\alpha - (\underline{\varepsilon} + \underline{p}) u e_\alpha \rfloor^* u \quad (45)$$

with the constant energy density and pressure defined by

$$\underline{\varepsilon} = (2\alpha_L \mathcal{T} - \beta - 6\mu\alpha_L \ell - 2\alpha_L^2 \ell^3 \varepsilon_0) \varepsilon_0, \quad (46)$$

$$\underline{p} = (2\alpha_L \mathcal{T} - \beta - 6\mu\alpha_L \ell - 2\alpha_L^2 \ell^3 \varepsilon_0) p_0. \quad (47)$$

The shifted “cosmological constant” reads:

$$\underline{\Lambda} = \Lambda^{\text{eff}} + \alpha_L \ell \left[ 2\mu(\varepsilon_0 - 2p_0) + \alpha_L \ell^2 \varepsilon_0^2 \right]. \quad (48)$$

As an example, let us derive the black hole type solution. We choose the local coordinates  $(t, r, \phi)$  and write the line element  $ds^2 = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$  with the Minkowski flat metric  $g_{\alpha\beta} = \text{diag}(-1, +1, +1)$  and the coframe

$$\vartheta^{\widehat{0}} = dt - h(r) d\phi, \quad \vartheta^{\widehat{1}} = \frac{1}{f(r)} dr, \quad \vartheta^{\widehat{2}} = f(r) r d\phi. \quad (49)$$

For the flow 2-form of the perfect fluid we assume the ansatz

$$u = \vartheta^{\widehat{1}} \wedge \vartheta^{\widehat{2}}. \quad (50)$$

Then from the differential condition (41), (42) we find

$$\frac{1}{r} \frac{dh}{dr} = 2\mu, \quad \text{or} \quad h(r) = \mu r^2 + h_0, \quad (51)$$

with some constant  $h_0$ . Substituting (49) into the effective Einstein equation (44), we find the second unknown function

$$f^2 = b_0 + b_1/r^2 + k r^2. \quad (52)$$

The constant energy density  $\varepsilon_0$  turns out to be an essential parameter which determines the other geometric and physical quantities of the solution. The constants  $b_0, b_1$  are arbitrary, whereas  $k$  and the pressure  $p_0$  are given by

$$k = \frac{2\alpha_L \ell \left[ \Lambda^{\text{eff}}(3\mu + \hat{\varepsilon}) - (\mu + \hat{\varepsilon})^3 \right] + (2\alpha_L \mathcal{T} - \beta) \left[ 3(\mu + \hat{\varepsilon})^2 - \Lambda^{\text{eff}} \right] - \ell \varepsilon_0 (2\alpha_L \mathcal{T} - \beta)^2}{4 [2\alpha_L \mathcal{T} - \beta - 2\alpha_L \ell (\mu + \hat{\varepsilon})]}, \quad (53)$$

$$p_0 = \frac{\mu^2 + \Lambda^{\text{eff}} + \hat{\varepsilon}(2\mu + \hat{\varepsilon})}{\ell [2\alpha_L \mathcal{T} - \beta - 2\alpha_L \ell (\mu + \hat{\varepsilon})]}. \quad (54)$$

Here we denoted  $\hat{\varepsilon} := \alpha_L \ell^2 \varepsilon_0$ . In the limit of the Einstein theory with both Chern-Simons terms absent,  $\theta_L = \theta_T = 0$ , the above quantities reduce to

$$k = \frac{3\chi\mu^2 - \Lambda - \ell\varepsilon_0}{4\chi}, \quad \ell p_0 = \chi\mu^2 + \Lambda. \quad (55)$$

In general, after a convenient rescaling of the time and the angular coordinates, we can write the above solution as

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{f^2} + g_{22} (d\phi + N^\phi dt)^2, \quad (56)$$

where (with the above constant parameters fixed as  $b_0 = -m, b_1 = J^2/4, h_0 = J/2$ )

$$f^2 = -m + (J/2r)^2 + k r^2, \quad N^2 = f^2 \left( 1 + \frac{r^2(k - \mu^2)}{\mu J - m} \right)^{-1}, \quad (57)$$

$$g_{22} = r^2 \left( 1 + \frac{r^2(k - \mu^2)}{\mu J - m} \right), \quad N^\phi = \left( \mu - \frac{J}{2r^2} \right) \left( 1 + \frac{r^2(k - \mu^2)}{\mu J - m} \right)^{-1}. \quad (58)$$

The metric (56) describes a black hole type configurations. The horizons are determined by

$$r_h^2 = \frac{m \pm \sqrt{m^2 - J^2 k}}{2k}. \quad (59)$$

Strictly speaking, this solution is no black hole because both the quasilocal angular momentum and the quasilocal mass [13,14] can be (one or both) divergent, in general. Accordingly, no definite finite mass and spin then can be attributed to such a configuration. In particular, let us compute the quasilocal angular momentum:

$$j(r) = \frac{f(g_{22})^{3/2}}{N} \frac{dN^\phi}{dr} = J + \frac{2r^2(\mu^2 - k)(\mu^2 r^2 - J)}{\mu J - m}. \quad (60)$$

This quantity obviously diverges for  $r \rightarrow \infty$ . The quasilocal mass is given by a more complicated expression:

$$m(r) = \frac{(m + \mathcal{O}(1/r^2)) \left( 1 + \frac{2r^2(k - \mu^2)}{\mu J - m} \right) - j(r) \left( \mu - \frac{J}{2r^2} \right)}{1 + \frac{r^2(k - \mu^2)}{\mu J - m}}. \quad (61)$$

Because of the last term in the numerator, this quantity also diverges for  $r \rightarrow \infty$ . There are, however, some particular cases when both quasilocal quantities yield the finite limits at the spatial infinity. The first case takes place for  $\mu = 0$ . Then we find the finite mass of the solution,  $M = m(\infty) = 2m$ . The quasilocal angular momentum is, however, still divergent unless  $J = 0$ . The resulting non-rotating configuration describes a nontrivial black hole with a horizon at  $r_h^2 = m/k$ . The second case happens when  $k = \mu^2$ . Then the resulting mass and spin are equal  $M = m$  and  $j(\infty) = J$ , respectively. In this case we recover the BTZ solution [17] which describes a black hole when  $m \geq \mu J$ , and a naked singularity for  $\mu J > m$ . It is instructive to compare this with the solution [15] in the topologically massive (DJT) gravity in three dimensions [16].

The curvature of the above solution is completely regular. We find for the anholonomic components of the Riemannian curvature 2-form

$$\widetilde{R}^{\widehat{0}\widehat{1}} = \mu^2 \vartheta^{\widehat{0}} \wedge \vartheta^{\widehat{1}}, \quad \widetilde{R}^{\widehat{0}\widehat{2}} = \mu^2 \vartheta^{\widehat{0}} \wedge \vartheta^{\widehat{2}}, \quad \widetilde{R}^{\widehat{1}\widehat{2}} = (4k - 3\mu^2) \vartheta^{\widehat{1}} \wedge \vartheta^{\widehat{2}}. \quad (62)$$

For  $k = \mu^2$  we recover the curvature of the anti-de Sitter spacetime, in complete agreement with the BTZ-limit.

## B. Cosmology with an ideal fluid

Let us analyze now the cosmological solutions. We choose the local coordinates  $(t, x, y)$  and write the line element  $ds^2 = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$  with the Minkowski flat metric  $g_{\alpha\beta} = \text{diag}(-1, +1, +1)$  and the coframe

$$\vartheta^{\widehat{0}} = dt, \quad \vartheta^{\widehat{1}} = R e^r dx, \quad \vartheta^{\widehat{2}} = R e^{-r} dy. \quad (63)$$

The two functions of the cosmological time,  $R = R(t)$  and  $r = r(t)$  describe the evolution of an anisotropic 3D universe. Accordingly, we assume that the energy density and the pressure depend on the time only:  $\varepsilon = \varepsilon(t)$ ,  $p = p(t)$ . Finally, we choose for the flow 2-form the ansatz  $u = \vartheta^{\widehat{1}} \wedge \vartheta^{\widehat{2}}$ , as above. The energy conservation (38) then yields the familiar equation:

$$\dot{\varepsilon} + 2\frac{\dot{R}}{R}(\varepsilon + p) = 0. \quad (64)$$

The dot denotes the time derivative in this subsection. Choosing now the equation of state  $p = \gamma\varepsilon$  (with constant  $\gamma$ ), we can integrate the above equation. This yields

$$\varepsilon = \frac{\varepsilon_0}{R^{2(1+\gamma)}}. \quad (65)$$

Substituting (37) and (63) into the effective Einstein equation (15), (17), we find the following system of ordinary differential equations:

$$-\frac{\dot{R}^2}{R^2} + \dot{r}^2 + \Lambda^{\text{eff}} = -\ell(2\alpha_L\mathcal{T} - \beta)\varepsilon + \alpha_L^2\ell^4\varepsilon^2, \quad (66)$$

$$-\frac{\ddot{R}}{R} + 2\frac{\dot{R}}{R}\dot{r} + \ddot{r} - \dot{r}^2 + \Lambda^{\text{eff}} = \ell(2\alpha_L\mathcal{T} - \beta)p - \alpha_L^2\ell^4\varepsilon(\varepsilon + 2p), \quad (67)$$

$$-\frac{\ddot{R}}{R} - 2\frac{\dot{R}}{R}\dot{r} - \ddot{r} - \dot{r}^2 + \Lambda^{\text{eff}} = \ell(2\alpha_L\mathcal{T} - \beta)p - \alpha_L^2\ell^4\varepsilon(\varepsilon + 2p), \quad (68)$$

$$2\alpha_L\ell\dot{r}(\varepsilon + p) = 0. \quad (69)$$

Comparing (67) and (68), we find

$$\dot{r} = \frac{c_0}{R^2}, \quad (70)$$

with an arbitrary integration constant  $c_0$ . Moreover, as we can immediately see, both equations (67) and (68) are redundant: They follow from (66) and the energy conservation (64). The further analysis is based on the simple observation that the equation (69) allows for only one of the following three cases: (i)  $\dot{r} = 0$ , (ii)  $\varepsilon = -p$ , or (iii)  $\alpha_L = 0$ . Let us consider these possibilities separately.

### 1. Case ( $\dot{r} = 0$ ): isotropic cosmology

When  $\dot{r} = 0$ , we can put  $r = 0$  without a loss of generality. Then using (65) in (66), we can integrate the latter equation for an arbitrary value of  $\gamma$ . The form of the solution depends on the sign of  $\Lambda^{\text{eff}}$  and on

$$\Delta = \frac{-\ell^2\varepsilon_0^2}{\chi^2 + 2\theta_T\theta_L}. \quad (71)$$

Let us at first assume that  $\lambda = \Lambda^{\text{eff}} > 0$ . Then for the different values of  $\Delta$  we find

$$R^{2(1+\gamma)} = \frac{1}{2\lambda} \left( -b + \sqrt{\Delta} \sinh[2\sqrt{\lambda}(1+\gamma)(t-t_0)] \right), \quad \Delta > 0, \quad (72)$$

$$R^{2(1+\gamma)} = \frac{1}{2\lambda} \left( -b + \sqrt{|\Delta|} \cosh[2\sqrt{\lambda}(1+\gamma)(t-t_0)] \right), \quad \Delta < 0, \quad (73)$$

$$R^{2(1+\gamma)} = \frac{1}{2\lambda} \left( -b + \exp[2\sqrt{\lambda}(1+\gamma)(t-t_0)] \right), \quad \Delta = 0. \quad (74)$$

Here  $t_0$  is an arbitrary integration constant and  $b := \ell(2\alpha_L \mathcal{T} - \beta) \varepsilon_0$ . The last case, (74), obviously describes the vacuum solution. The corresponding spacetime is a 3D de Sitter manifold. The solutions (72) and (73) describe asymptotically (when  $t \rightarrow \pm\infty$ ) de Sitter spacetimes.

For the negative effective cosmological constant,  $\lambda = -|\Lambda^{\text{eff}}| < 0$ , solution exists only for  $\Delta < 0$  and reads:

$$R^{2(1+\gamma)} = \frac{1}{2|\lambda|} \left( b + \sqrt{|\Delta|} \sin[2\sqrt{|\lambda|}(1+\gamma)(t-t_0)] \right). \quad (75)$$

Finally, for the vanishing effective cosmological constant  $\Lambda^{\text{eff}} = 0$ , we find the solution

$$R^{2(1+\gamma)} = \alpha_L^2 \ell^4 \varepsilon_0^2 / b + b(1+\gamma)^2 (t-t_0)^2. \quad (76)$$

## 2. Case ( $\varepsilon = -p$ ): anisotropic de Sitter

For the vacuum equation of state we have  $\gamma = -1$  and (65) yields constant energy density  $\varepsilon = \varepsilon_0$ . Using this, and substituting (70) into (66), we can integrate the resulting equation. This yields

$$R^2 = \frac{c_0}{\sqrt{Q}} \sinh \left[ 2\sqrt{Q}(t-t_0) \right]. \quad (77)$$

Here  $t_0$  is again an integration constant, and we denoted

$$Q := \Lambda^{\text{eff}} + \ell(2\alpha_L \mathcal{T} - \beta) \varepsilon_0 - \alpha_L^2 \ell^4 \varepsilon_0^2. \quad (78)$$

Using (77) in (70), we find the second unknown function

$$r = \frac{1}{2} \ln \tanh \left[ \sqrt{Q}(t - t_0) \right], \quad (79)$$

and thus finally the line elements reads

$$ds^2 = -dt^2 + \sinh^2 \left[ \sqrt{Q}(t - t_0) \right] dx^2 + \cosh^2 \left[ \sqrt{Q}(t - t_0) \right] dy^2. \quad (80)$$

The resulting geometry has the Riemannian curvature of the de Sitter 3D spacetime  $\tilde{R}^{\alpha\beta} = -Q \vartheta^\alpha \wedge \vartheta^\beta$ . The above was derived under the assumption that  $Q > 0$ . In case it happens that  $Q < 0$ , we find instead of (77):

$$R^2 = \frac{c_0}{\sqrt{|Q|}} \sin \left[ 2\sqrt{|Q|}(t - t_0) \right], \quad (81)$$

and accordingly the line element is changed to

$$ds^2 = -dt^2 + \sin^2 \left[ \sqrt{|Q|}(t - t_0) \right] dx^2 + \cos^2 \left[ \sqrt{|Q|}(t - t_0) \right] dy^2. \quad (82)$$

This metric describes the anti-de Sitter spacetime with the curvature  $\tilde{R}^{\alpha\beta} = |Q| \vartheta^\alpha \wedge \vartheta^\beta$ .

Finally, in case when  $Q = 0$ , we find  $R^2 = 2c_0(t - t_0)$ , and  $2r = \ln(t - t_0)$  which yields the flat spacetime geometry  $ds^2 = -dt^2 + (t - t_0)^2 dx^2 + dy^2$ .

### 3. Case ( $\alpha_L = 0$ ): Heckmann-Schücking type solutions

As we saw in Sec. II, when  $\alpha_L = \theta_L = 0$ , our effective theory in fact reduces to the Einstein gravity in three dimensions. This particular case yields additional solutions to the system (66)-(69) which are analogs of the generalized Heckmann-Schücking cosmological solutions obtained in four dimensions [18,19]. As in [19], we choose the matter as the mixture of the three media - vacuum fluid, dust, and the stiff matter. Accordingly, the energy density then reads as the sum of the three terms, representing vacuum, dust and stiff matter, respectively:

$$\varepsilon = \varepsilon_0 + \frac{\varepsilon_1}{R^2} + \frac{\varepsilon_2}{R^4}, \quad (83)$$

with the positive integration constants  $\varepsilon_0, \varepsilon_1, \varepsilon_2$ . After substituting this into (66), the integration of (66) and (70) is straightforward. The result essentially depends on the parameter (78) which in the absence of the Lorentz Chern-Simons coupling ( $\alpha_L = 0$ ) reduces to



$Q = \Lambda^{\text{eff}} + \ell\varepsilon_0/\chi$  (note that  $\beta = -1/\chi$  then). The explicit solution of (66) and (70) can be conveniently written in the form

$$R = R_1 R_2, \quad 2r = \frac{c_0}{\sqrt{c_0^2 + \ell\varepsilon_2/\chi}} \ln \frac{R_1}{R_2}, \quad (84)$$

where the functions  $R_1(t)$  and  $R_2(t)$  are determined by:

$$R_1 = \begin{cases} \frac{1}{\sqrt{Q}} \sinh(\sqrt{Q}t), & \text{if } Q > 0, \\ \frac{1}{\sqrt{|Q|}} \sin(\sqrt{|Q|}t), & \text{if } Q < 0, \\ t, & \text{if } Q = 0. \end{cases} \quad (85)$$

$$R_2 = \begin{cases} \frac{\ell\varepsilon_1}{\chi\sqrt{Q}} \sinh(\sqrt{Q}t) + 2\sqrt{c_0^2 + \ell\varepsilon_2/\chi} \cosh(\sqrt{Q}t), & \text{if } Q > 0, \\ \frac{\ell\varepsilon_1}{\chi\sqrt{|Q|}} \sin(\sqrt{|Q|}t) + 2\sqrt{c_0^2 + \ell\varepsilon_2/\chi} \cos(\sqrt{|Q|}t), & \text{if } Q < 0, \\ \frac{\ell\varepsilon_1}{\chi} t + 2\sqrt{c_0^2 + \ell\varepsilon_2/\chi}, & \text{if } Q = 0. \end{cases} \quad (86)$$

As a result, the metric takes the form of the Heckmann-Schücking type anisotropic cosmological solution:

$$ds^2 = -dt^2 + \left(R_1^{p_1} R_2^{1-p_1}\right)^2 dx^2 + \left(R_1^{p_2} R_2^{1-p_2}\right)^2 dy^2. \quad (87)$$

Here we introduce the two constant parameters which satisfy the Kasner type conditions:

$$p_1 + p_2 = 1, \quad p_1^2 + p_2^2 = 1 - \xi/2. \quad (88)$$

Like in four dimensions [19], the “non-Kasner” parameter

$$\xi := \frac{\ell\varepsilon_2}{\chi c_0^2 + \ell\varepsilon_2} \quad (89)$$

vanishes when the stiff matter component is absent.

The corresponding Riemannian curvature 2-form reads:

$$\tilde{R}^{\widehat{0}\widehat{1}} = \left(-Q + \frac{4p_1 p_2 (c_0^2 + \ell\varepsilon_2/\chi)}{R^2}\right) \vartheta^{\widehat{0}} \wedge \vartheta^{\widehat{1}}, \quad (90)$$

$$\tilde{R}^{\widehat{0}\widehat{2}} = \left(-Q + \frac{4p_1 p_2 (c_0^2 + \ell\varepsilon_2/\chi)}{R^2}\right) \vartheta^{\widehat{0}} \wedge \vartheta^{\widehat{2}}, \quad (91)$$

$$\tilde{R}^{\widehat{1}\widehat{2}} = \left(-Q - \frac{4p_1 p_2 (c_0^2 + \ell\varepsilon_2/\chi)}{R^2} - \frac{\ell\varepsilon_1/\chi}{R}\right) \vartheta^{\widehat{1}} \wedge \vartheta^{\widehat{2}}. \quad (92)$$

In the absence of the dust and the stiff matter contributions (i.e., when both  $\varepsilon_1 = 0$  and  $\varepsilon_2 = 0$ ), we recover the anisotropic de Sitter geometry studied in the previous subsection.

## V. CONCLUSIONS

In this paper, we have derived the effective 3D Einstein theory which is equivalent to the Mielke-Baekler model for all matter sources with the vanishing dynamical spin current. The Lagrangian of the model includes the standard Hilbert-Einstein term plus the translational and the rotational Chern-Simons terms. It is shown that the purely Chern-Simons gravitational theory (when the Hilbert-Einstein term is absent) are only consistent in vacuum, whereas the nontrivial energy-momentum current leads, in general, to the inconsistencies.

The general formalism is applied to the cases of electromagnetic and the perfect fluid sources. The new exact solutions of the 3D gravity theory are obtained: the gravitational  $pp$ -waves for the Maxwell source, and the black hole type solutions and the anisotropic cosmological solutions for the perfect fluid matter.

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